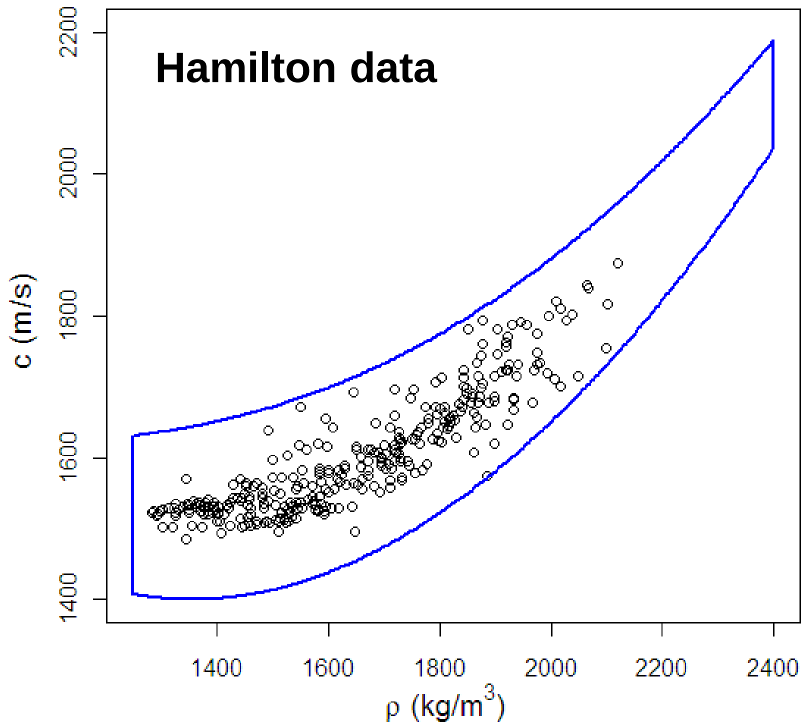
# Uncertainty Estimation: Overview

1. Prior information
2. Model selection
3. Data misfit
4. Parameter estimation
5. Uncertainty estimation
6. Uncertainty/variability
7. Joint Inversion

# 1. Prior Information

- Quantitative information applied to inversion independent of measured data
- **Explicit:**
  - Parameter bounds (bounded uniform distribution)
  - Non-uniform prior distributions
  - Inter-parameter relationships
- **Implicit:**
  - Physics models and parameterizations considered



# Prior Information

- Prior information (particularly parameterization, hard bounds) can strongly influence solution
  - Important to specify priors in comparing uncertainty results

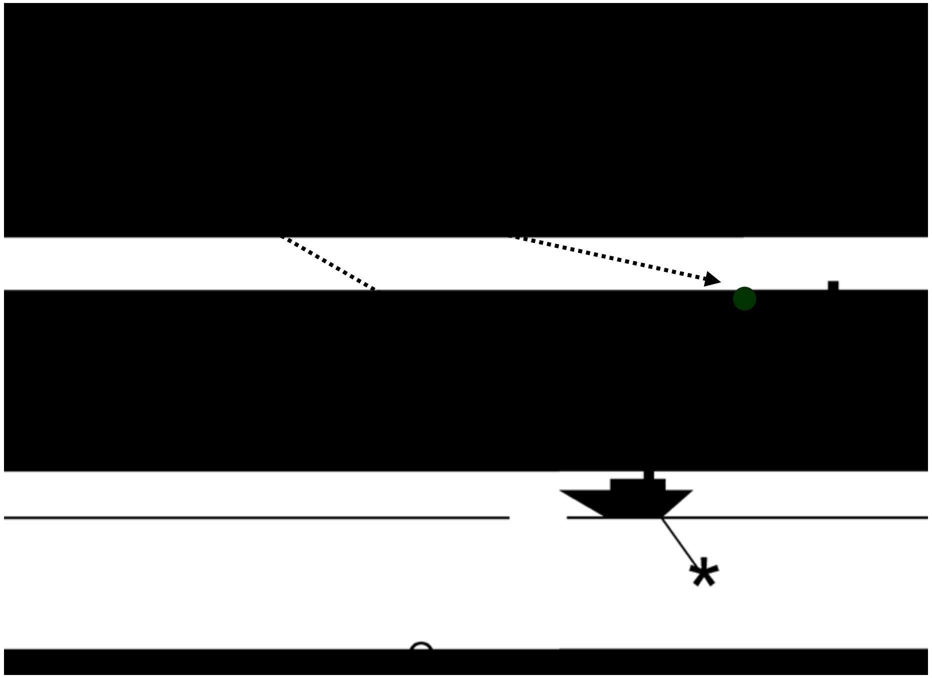
- Common goal:
  - Constrain parameters to physically-reasonable values
  - Allow data information to primarily determine solution

- If data and prior disagree:
  - Reassess data and error estimates
  - Reassess prior, including physics model and parameterization

# 2. Model Selection

- **Physics model**
  - Fluid, elastic or poro-elastic?
  - Range independent/dependent?
  - Plane wave or spherical wave?

- **Model parameterization**
  - Number of layers/segments?



**n layers—best choice of n?**

# Model Selection

- Quantitative uncertainty estimation requires appropriate model parameterization
  - **Under-parameterization** can lead to under-fitting data, biased parameter estimates, under-estimated uncertainties
  - **Over-parameterization** can lead to over-fitting data, unconstrained structure, over-estimated uncertainties
- Seek simplest parameterization consistent with resolving power of the data

# Model Selection

- Qualitative Model Selection:
  - Based on insight and experience
- Quantitative Model Selection:
  - **Bayesian information criterion** (BIC)—point estimate based on optimization that balances data fit and number of parameters
  - **Evidence**—Integral estimate of parameterization likelihood given the data, based on sampling
  - **Trans-dimensional inversion**
  - **Multiple-model particle filter**

Include number of parameters as unknown in inversion

# Example: BIC

- Invert Scholte (interface) wave dispersion curves from ambient noise
  - Invert fundamental mode only
  - Invert first 3 modes

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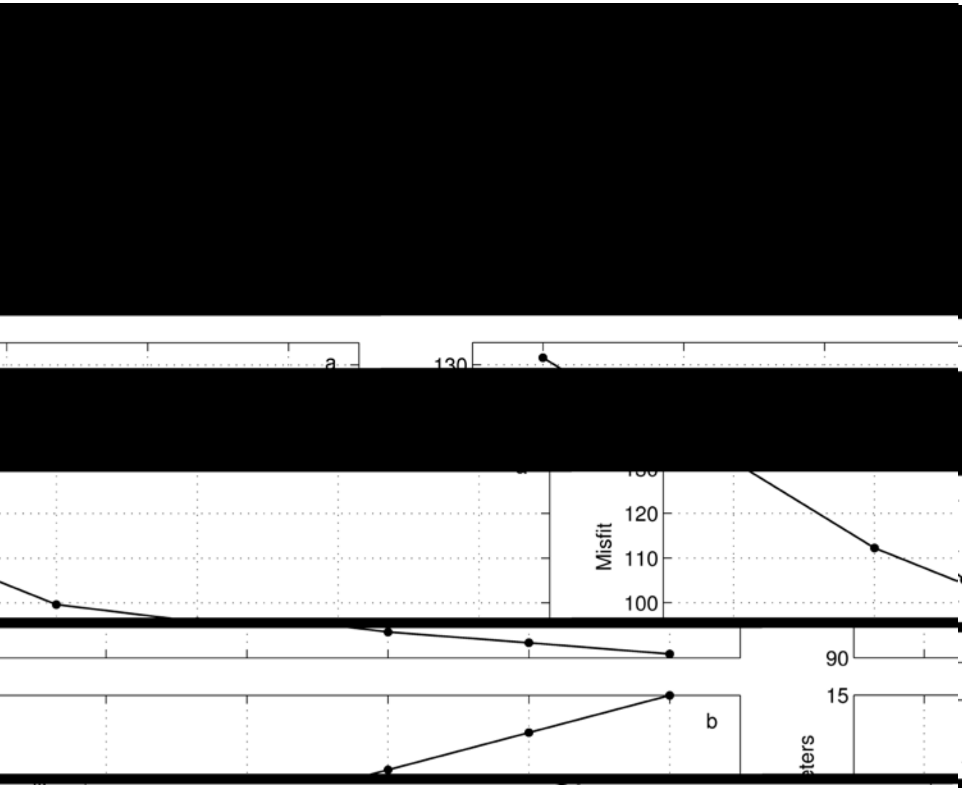
# BIC: 1 & 3 Mode Inversions



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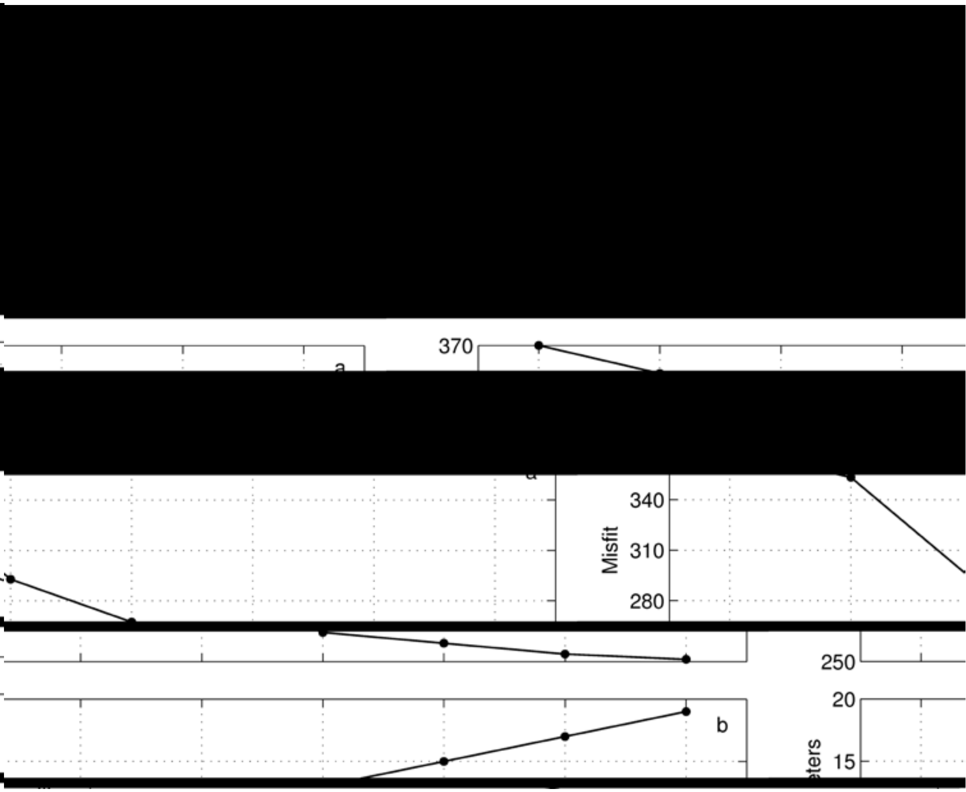

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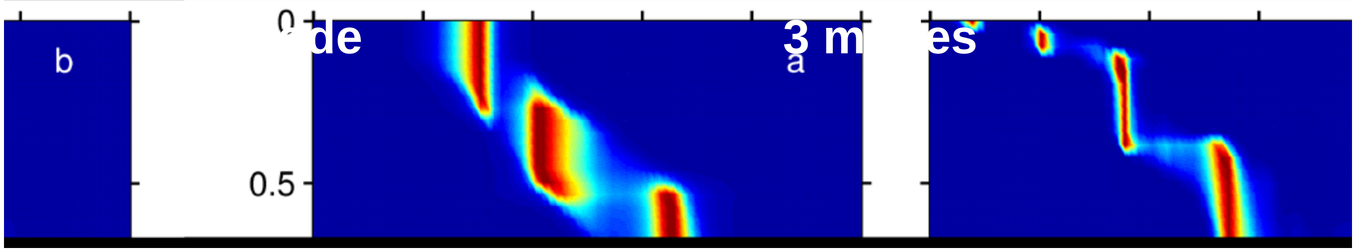
**1 mode: 5 layers resolved**

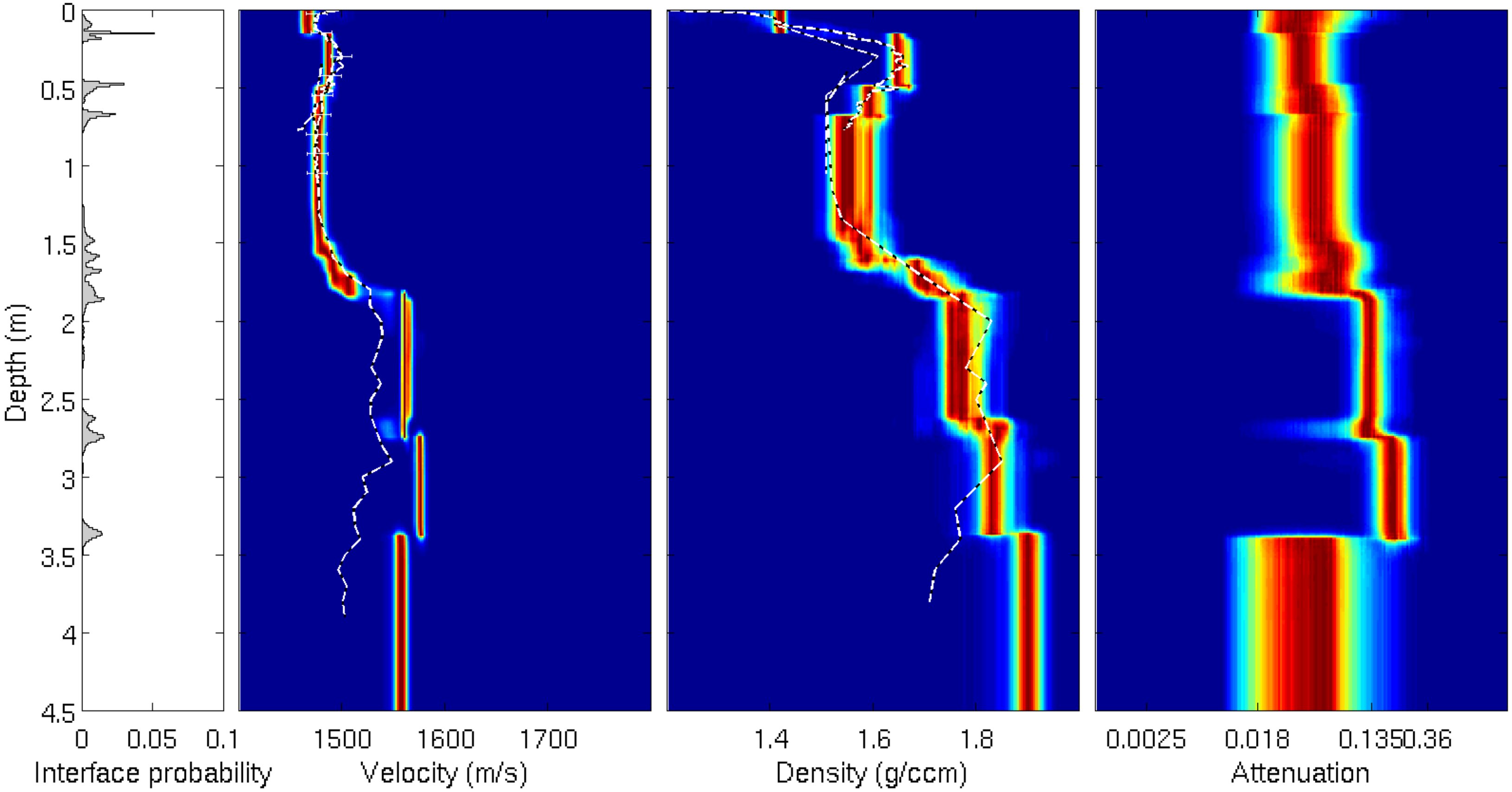
**3 modes: 8 layers resolved**

# MAP Profiles

# Marginal Probability Profiles



# Trans-D Reflection Inversion



# 3. Data Misfit Function

- Misfit quantifies difference between measured and modeled data
- Parameter estimation (optimization):
  - Minimize any reasonable misfit function; result is corresponding best-fit model according
  - Likelihood-based misfit provides *efficient* estimator
- Uncertainty estimation:
  - Generally requires likelihood-based misfit
  - Maxent methods can specify least-informative misfit function for a given constraint

# Likelihood Function

- **Likelihood:** Interprets data uncertainty distribution as a function of model parameters
  - Consistent with inversion as mapping data uncertainty distribution (data space) to parameter uncertainty distribution (model space)
- Requires estimating data uncertainties (measurement and theory errors)
  - Form of distribution (Gaussian, Laplace, …)
  - Statistical properties (variance, covariance) estimated from data residuals or included as unknowns in inversion

# Examples

- IID Gaussian data errors

$$P(\mathbf{d},\mathbf{m}) = \frac{1}{\left(2\pi\sigma^2\right)^{N/2}} \exp\left[ \underbrace{-\left|\mathbf{d}-\mathbf{d}(\mathbf{m})\right|^2 / 2\sigma^2}_{\substack{\text{misfit } E(\mathbf{m}) \\ \text{(least squares)}}} \right]$$

- IID Gaussian errors, unknown source strength

$$E(\mathbf{m}) = \left[ \left|\mathbf{d}\right|^2 - \frac{\left|\mathbf{d}^T \mathbf{d}(\mathbf{m})\right|^2}{\left|\mathbf{d}(\mathbf{m})\right|^2} \right] / \sigma^2$$

(Bartlett processor)

# Data Errors

- Specifying likelihood requires quantifying the data error distribution

- Data errors = Inability to model measured data:
  - **Measurement errors:** ambient noise, instrumental uncertainties, etc.
  - **Theory errors:** due to idealized physics and simplified parameterization, etc.

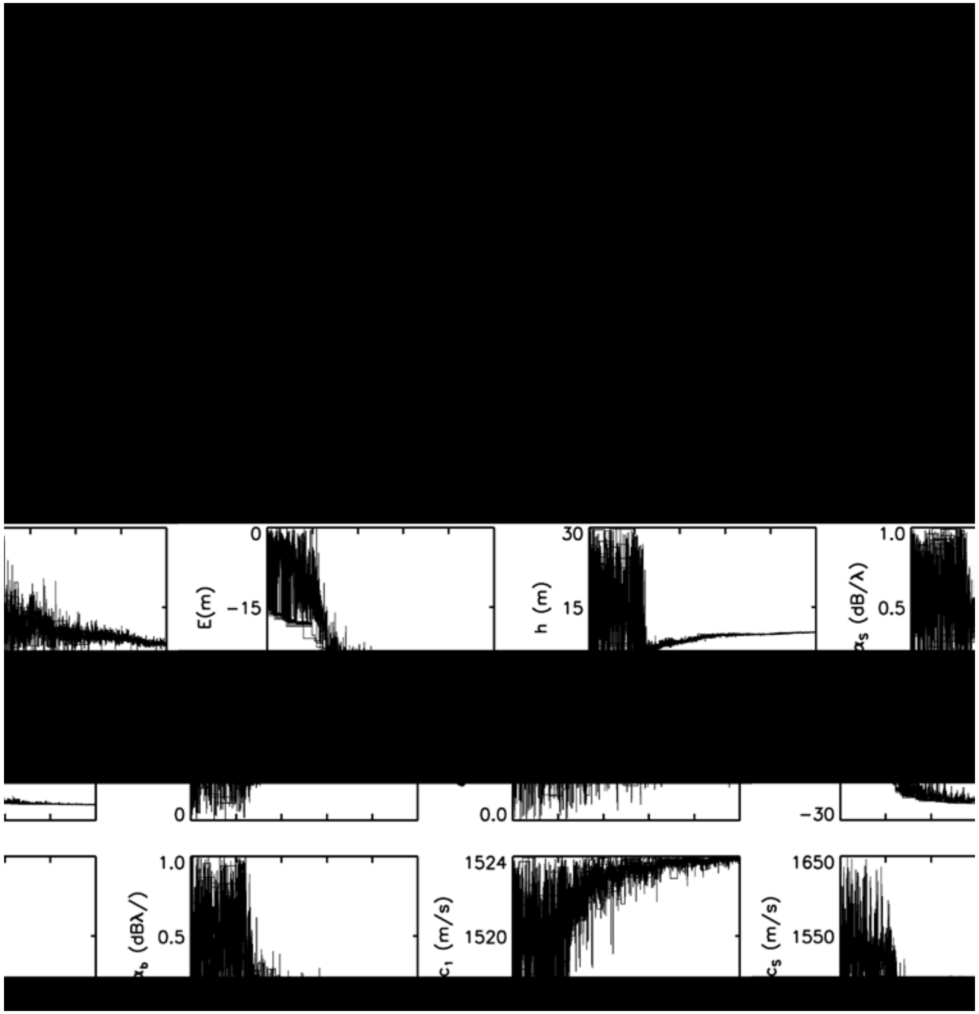
- Ensure modeling is as accurate as possible and data sample over error processes (difficult)
  - Sample over noise, internal waves, variability, etc.
  - Collect multiple data sets (same & different types)
  - Note: beyond a point, denser data lead to correlated errors

# 4. Parameter Estimation

- Minimize data misfit via optimization
  - ➢ Linearized inversion (prone to local minima)
  - ➢ Global search
  - ➢ Hybrid optimization

➢ Repeat optimization to ensure stable result

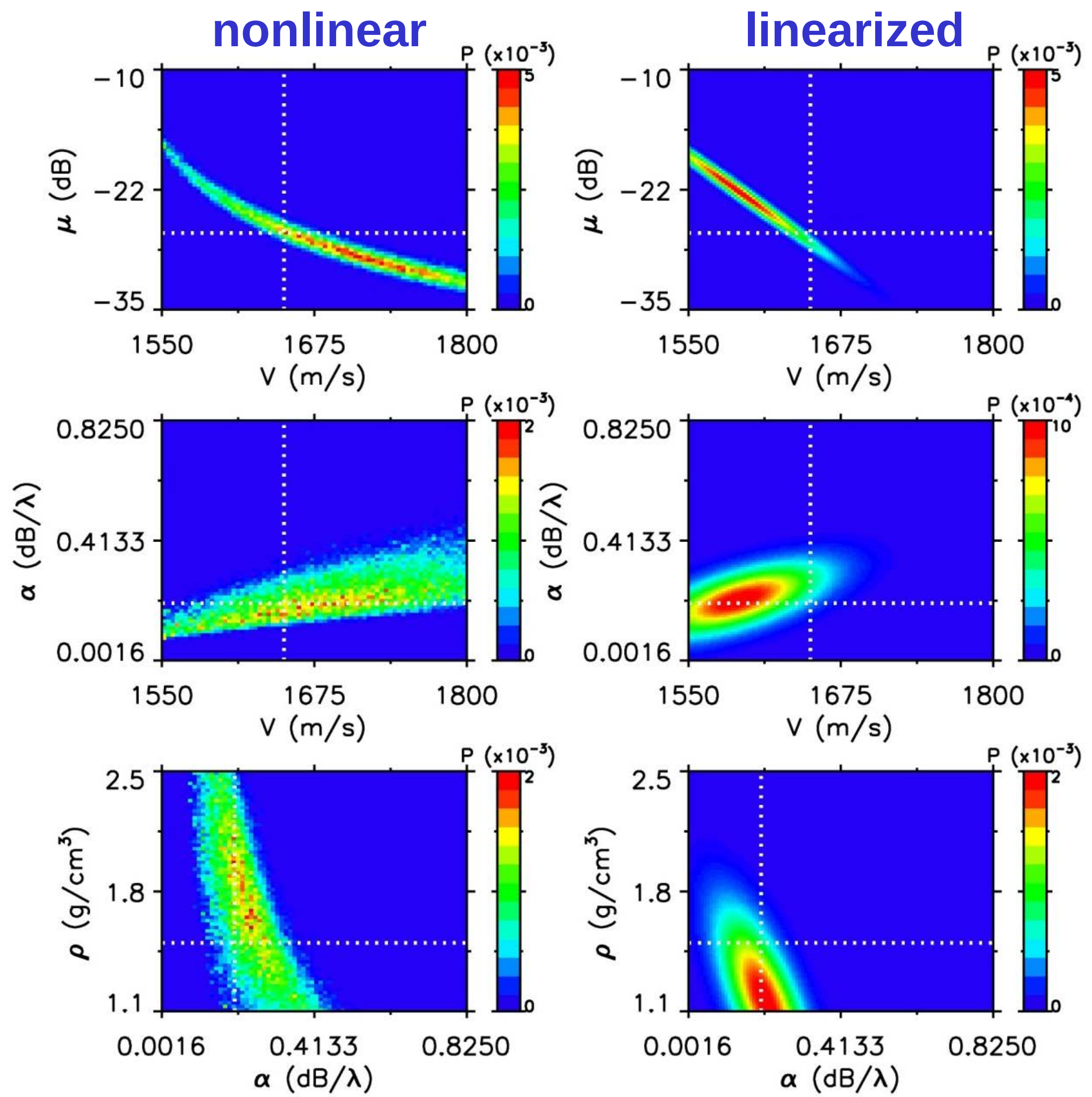
➢ Mean model via sampling



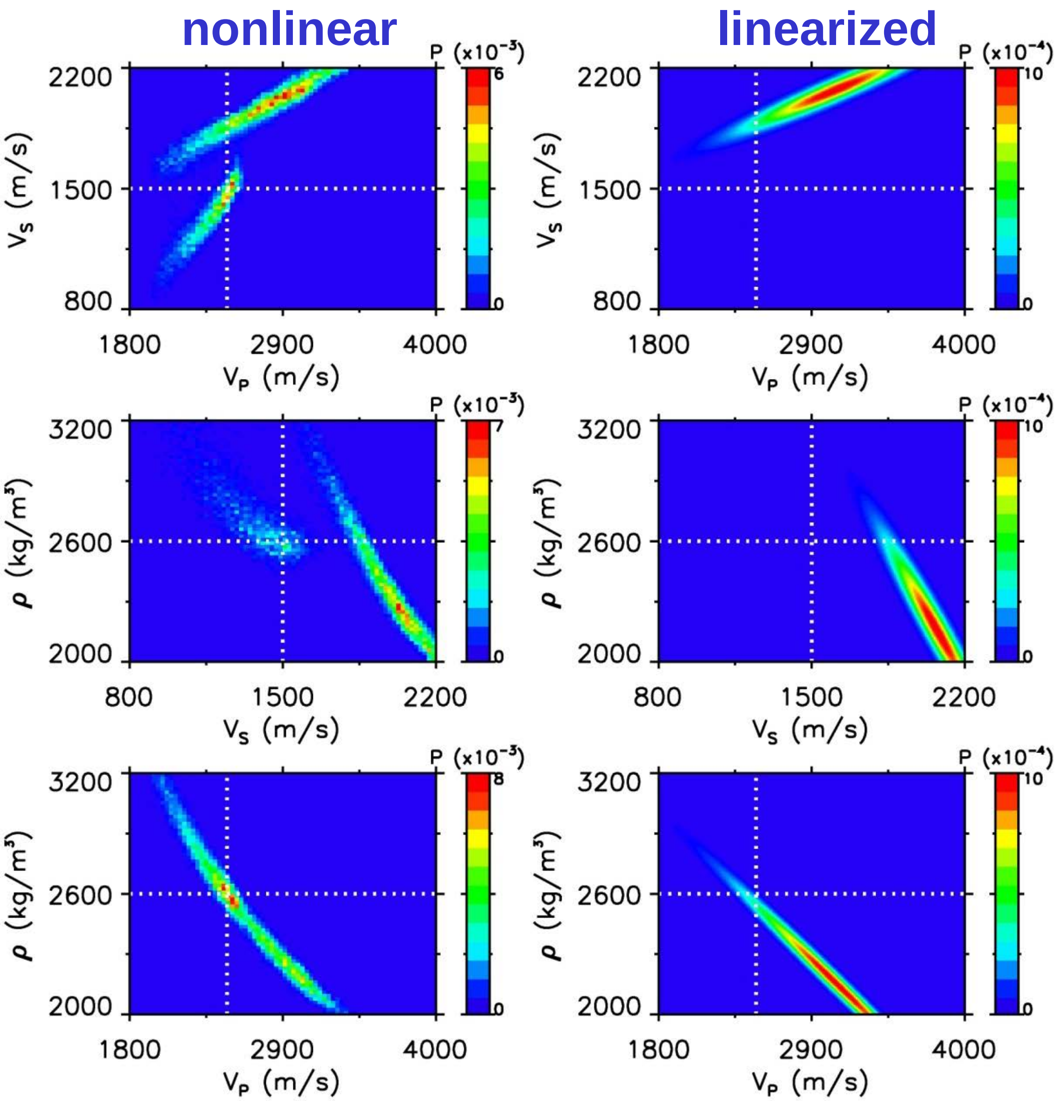
# 5. Uncertainty Estimation

- **Linearization**—Analytic result (exact solution to approximate problem)
  - Gaussian data uncertainties and unbounded-uniform or Gaussian prior leads to Gaussian parameter uncertainties
  - Efficient, potentially inaccurate
- **Nonlinear**—Numerical sampling (approx solution to exact problem)
  - Monte Carlo/Importance sampling
  - Markov-chain Monte Carlo (Metropolis Hastings, Gibbs sampling)
  - Parallel-tempering
  - Numerically intensive; sampling/convergence issues

# Joint Uncertainties—Reverb

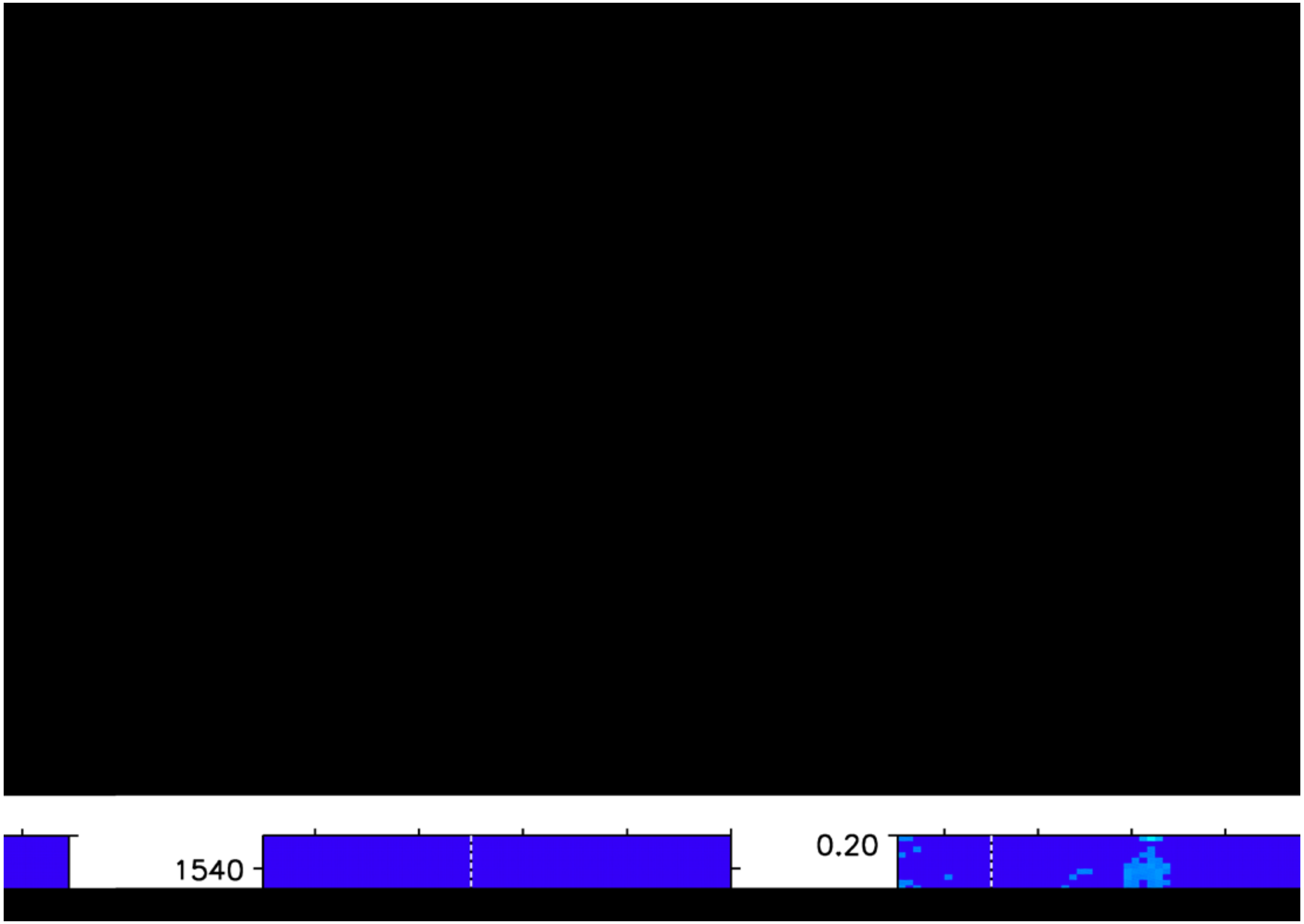


# Joint Uncertainties—Reflection



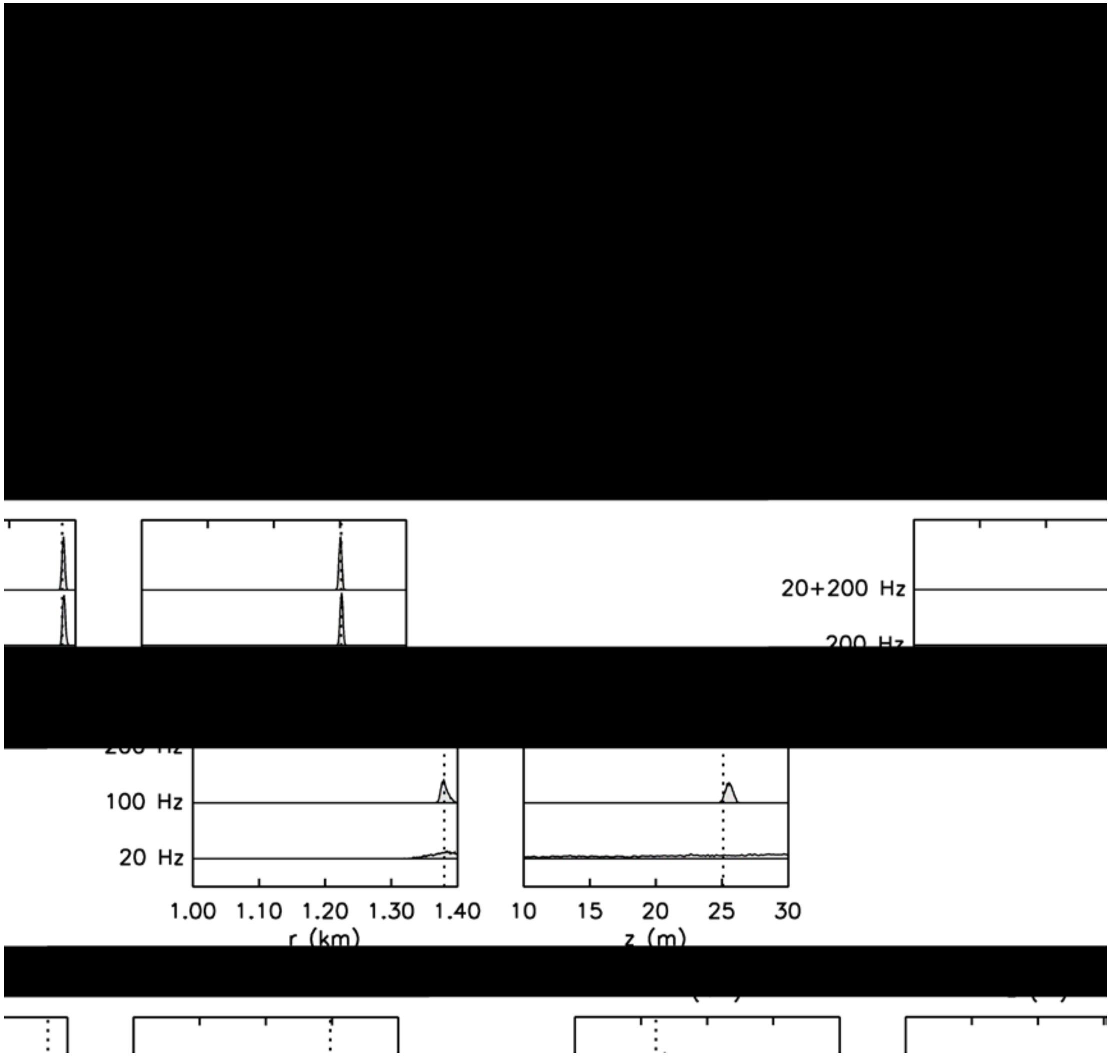
**nonlinear** **linearized**

# Uncertainties—Reverb/Scattering



# Experiment Planning: Simulation

- Uncertainty estimation for simulations quantifies ideal sensitivity and can help plan experiment factors



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Example: Frequencies in MFI

# 6. Variability & Uncertainty

- **Variability**
  - Measure of inherent spatial or temporal heterogeneity in an environmental property
  - Ideally quantified statistically/probabilistically
  - Intrinsic property of the environment—cannot be reduced by improved experiment or data analysis, although these can improve variability estimates
- **Uncertainty**
  - Measure of knowledge of an environmental parameter
  - Ideally quantified statistically/probabilistically
  - Property of environmental knowledge, not of the environment itself—can be reduced by improved experiments or data analysis

# Variability & Uncertainty

- Inversion uncertainties quantify accuracy of the model parameter estimates adopted to represent the environment

- Consider a parameter (e.g., sound speed of upper layer) over an experimental footprint
    - Uncertainty quantifies accuracy of average sound speed over footprint
    - Uncertainty does not quantify sound-speed variability over footprint (accurate average could be obtained for a highly variable property)
    - Parameter estimates involve non-uniform averaging so care required in interpretation

# Variability & Uncertainty

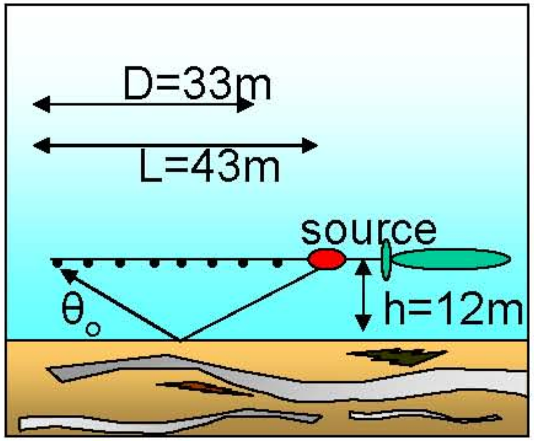
- Variability & Uncertainty are distinct but related
  - Variability can cause theory/modeling errors which lead to parameter uncertainties
  - If theory errors due to variability dominate and are adequately sampled, uncertainty estimates can quantify variability (care required)

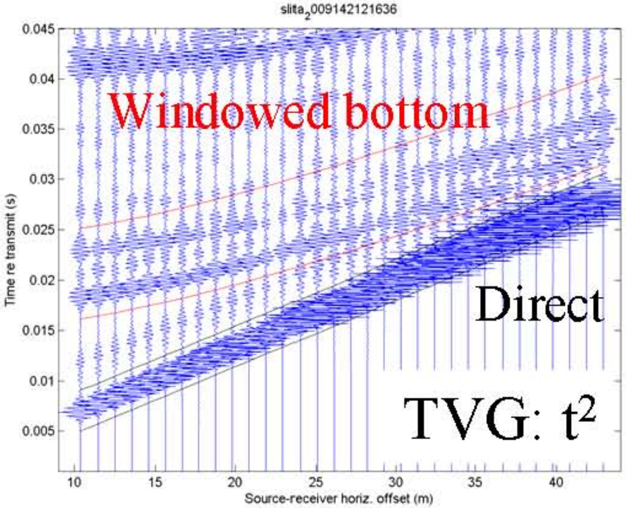
# Variability & Uncertainty

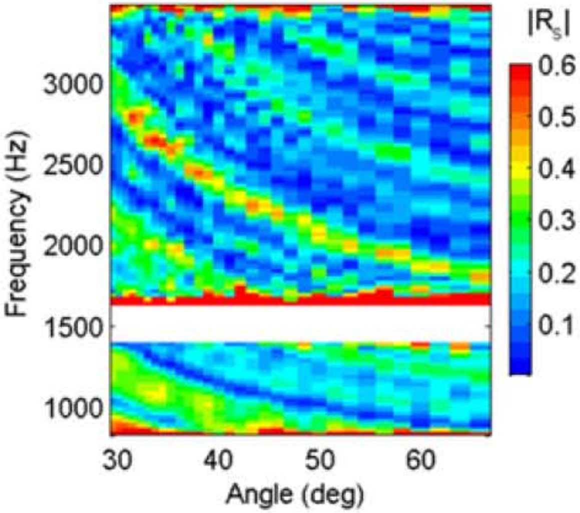
- Variability study:
  - Localized, high-resolution measurements closely spaced in space or time
  - Significant differences between recovered parameters represent variability
  - Uncertainty estimation essential to determine if observed differences due to environmental variability or uncertain parameter estimates

# Sequential Trans-D Inversion

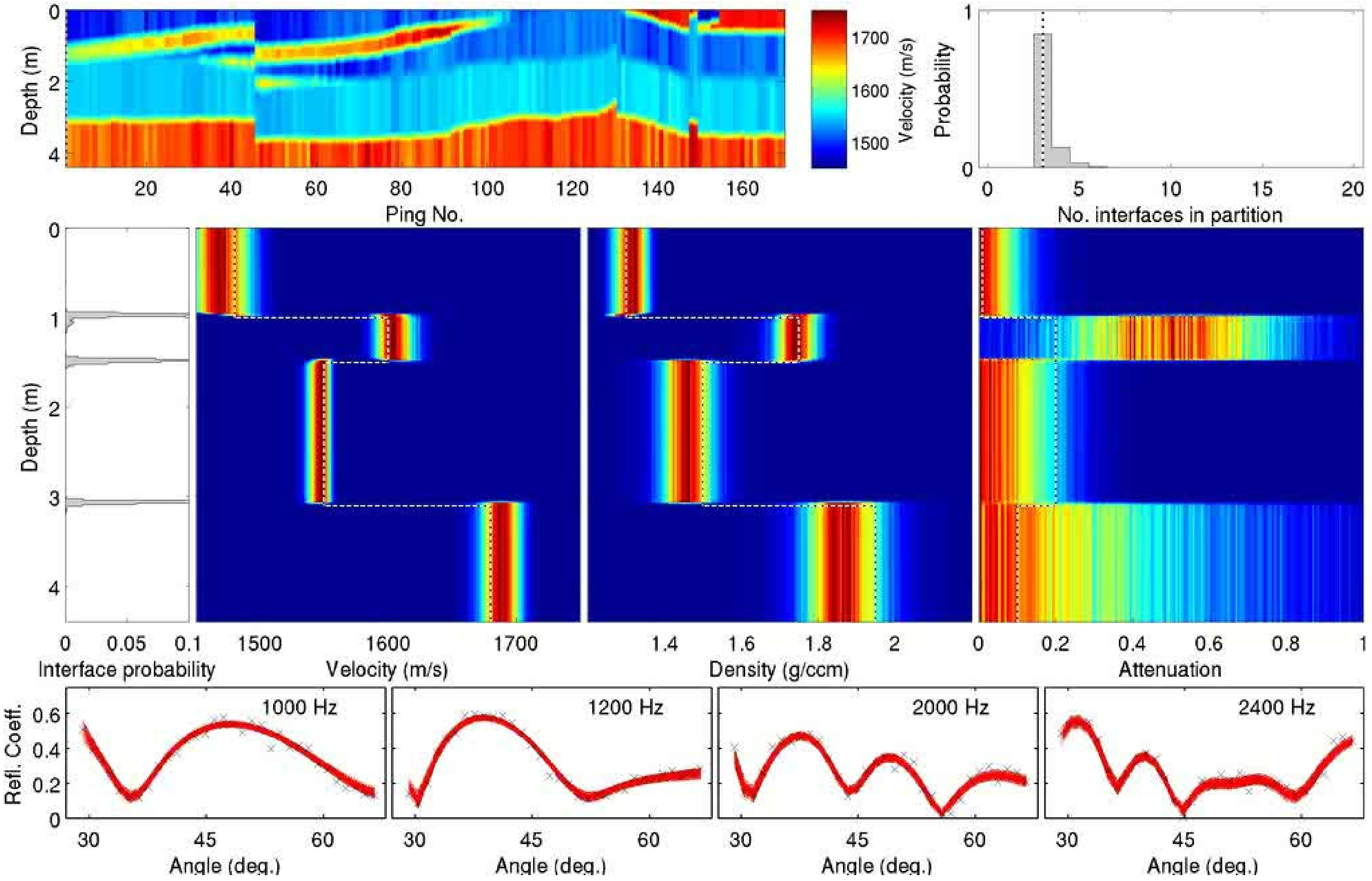
- **AUV-towed source and array:**
  - Reflection data for small seafloor footprint
  - Mobile system for sub-bottom mapping
  - Reduces effects of seabed/ocean variability







# Sequential Trans-D Inversion



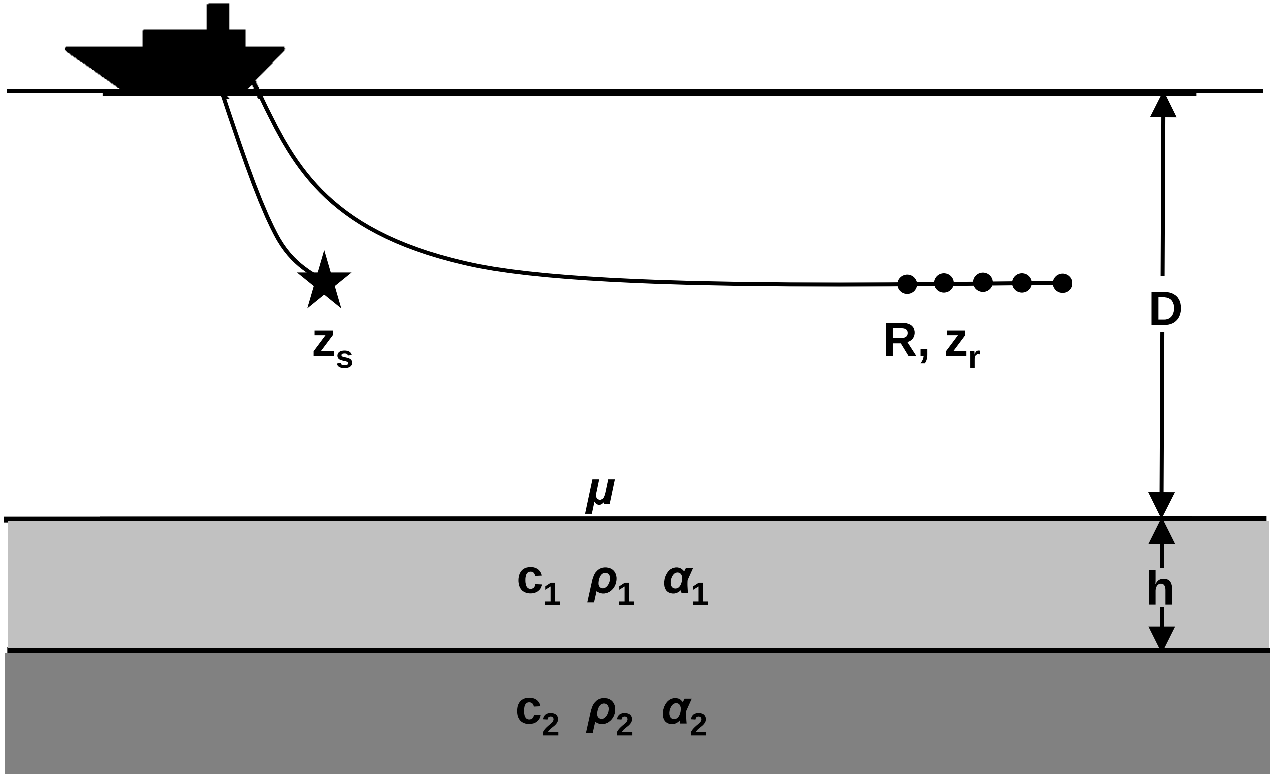
# 7. Joint Inversion

- Joint (simultaneous) inversion of different data brings more information to bear

- Different physics for different data can overcome
  - Low sensitivity to some parameters
  - Inter-parameter correlations

# Example: Reverb/Prop Inversion

- Invert (separately and jointly):
  - Short-range propagation data
  - Reverb data



# Reverb Inversion—Joint Marginals

- Strong inter-parameter correlations from reverb physics

# Propagation—Joint Marginals

- Different parameter correlations arise from different physics

# Reverb + Propagation Inversion

- Geoacoustics & scattering well resolved

# Inversion Comparison

Reverb

Prop

Reverb
+ Prop

Reverb

Prop

Reverb
+ Prop